ANALYSIS OF THE UTILITY OF LAPLACE TRANSFORMS FOR NON-LINEAR DIFFERENTIAL EQUATIONS

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THESIS

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Analysis of the Utility of Laplace Transforms For Non-Linear Differential Equations

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ABSTRACT

The use of Laplace Transforms has long been one of the primary methods utilized in solving linear differential equations but recently the extension and application of this method to the solution of basic non-linear systems has been proposed. This latter technique is presented and expanded to include various non-linear functions. The applicability of this method to systems of varied form and complexity is then explored and the validity of the derived solution investigated.

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I. INTRODUCTION

Laplace Transforms have been typically applicable and limited to linear systems. However, Baycura [Ref. 1 and 2] in two recent papers has proposed an extension of the Laplace Transform method to non-linear systems. By expanding the Laplace integral into an infinite series and integrating by parts n - times an infinite series was developed for non-linear terms. Through a further correlation of these derived series with series of linear transforms the transform for a non-linear term was obtained. These papers develop this technique for higher-order derivatives and powers of the dependent variables illustrated with several applicable examples.

The purpose of this paper will be to present the general method as developed by Baycura and to extend this method to varied and more complicated equations of non-linear systems. The latter will be accomplished by developing the transforms for the product of derivatives of the dependent variable and the dependent variable raised to the n^{th} power, and derivatives raised to the n^{th} power. The validity of the transforms will then be shown in several applicable examples. In the examples a computer method designed to find the inverse Laplace Transform will be introduced and explained.

II. LAPLACE TRANSFORM CONCEPT

A. THE METHOD OF INTEGRATING BY PARTS

The following generally summarizes Baycura's method of utilizing Laplace Transforms on non-linear systems using the method of integration by parts. Considering the product of two time-dependent functions, x(t) and y(t), which are to be integrated using integration by parts n - times, obtain the general formula:

$$\int_0^\infty xydt = x \int_0^\infty ydt - \dot{x} \int_0^\infty\int ydtdt + \ddot{x} \int_0^\infty\int\int ydtdtdt - \dots +$$

$$(-1)^{n-1} x^{(n-1)} \int_0^\infty \left(\int y\, dt^{n-1} \right) dt + \dots \qquad (2\text{-}1)$$

$n = 1, 2, 3, \dots$

where $x^{(n-1)}$ indicates the $(n - 1)^{th}$ derivative of x.

Applying this general formula to the Laplace Integral, equation (2-1) can be used to find the series form of the Laplace Transform of a function x(t) allowing $y(t) = e^{-st}$. This integration gives the series form of:

$$\int_0^\infty xe^{-st} = x \int_0^\infty e^{-st}\, dt - \dot{x} \int_0^\infty\int e^{-st}\, dtdt + \ddot{x} \int_0^\infty\int\int e^{-st} dt\, dt\, dt - \dots \qquad (2\text{-}2)$$

Evaluating at the upper limit it is found that the value of the integral approaches zero as t approaches infinity. Then evaluating at the lower limit the integral becomes:

$$\int_0^\infty xe^{-st} = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \dots \qquad (2\text{-}3)$$

or

$$\mathcal{L}[x(t)] = x(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots . \qquad (2\text{-}4)$$

B. APPLICATION TO DERIVATIVES

Equation (2-4) depicts the Laplace Transform of x(t) in an infinite series expansion involving the initial conditions of the function. Through similiar manipulations the transforms of the derivatives may be obtained:

$$\mathcal{L}[\dot{x}(t)] = \int_0^\infty \dot{x}\, e^{-st}\, dt = \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \frac{\dddot{x}(0)}{s^3} + \ldots .$$

$$= sX(s) - x(0) \qquad (2\text{-}5)$$

$$\mathcal{L}[\ddot{x}(t)] = \int_0^\infty \ddot{x}\, e^{-st}\, dt = \frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \ldots$$

$$= s^2 X(s) - s\,x(0) - \dot{x}(0) \qquad (2\text{-}6)$$

C. APPLICATION TO NON-LINEAR FUNCTIONS

The simplest non-linear function to investigate is $x^2(t)$. Using the same technique, expansion by parts yields:

$$\mathcal{L}[x^2(t)] = x\int_0^\infty x e^{-st}\, dt - \dot{x}\int_0^\infty\!\!\int x e^{-st}\, dt\, dt + \ddot{x}\int_0^\infty\!\!\int\!\!\int x e^{-st}\, dt\, dt\, dt - \ldots \qquad (2\text{-}7)$$

Taking each of the above terms individually and again expanding by parts obtain:

$$x\int_0^\infty x e^{-st}\, dt = x(0)\, X(s) \qquad (2\text{-}8)$$

$$-\dot{x}\int_0^\infty\!\!\int x e^{-st}\, dt\, dt = \frac{\dot{x}(0)\ X(s)}{s} \qquad (2\text{-}9)$$

$$\ddot{x} \int_0^\infty \int\int x e^{-st} \, dt \, dt \, dt = \frac{\ddot{x}(0) \; X(s)}{s^2} \tag{2-10}$$

The summation of these terms then yields:

$$\mathcal{L}\left[x^2(t)\right] = \left[x(0) + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \ldots . \right] X(s) = s\,X^2(s) \tag{2-11}$$

Through similiar computations, transforms of higher-order functions may be obtained. The transform of the function cubed would be:

$$\mathcal{L}\left[x^3(t)\right] = \left[x(0) + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \ldots \right] s\,X(s) = s^2 X^3(s) \tag{2-12}$$

Generalizing it can be stated that:

$$\mathcal{L}\left[x^n(t)\right] = s^{n-1} \, X^n(s) \tag{2-13}$$

III. LAPLACE TRANSFORMS FOR NON-LINEAR FUNCTIONS

A. TRANSFORMS FOR THE PRODUCT OF A FUNCTION AND ITS FIRST DERIVATIVE

Transforming $\dot{x}(t)\,x(t)$ by applying integration by parts the following derivation is obtained:

$$\mathcal{L}\,[\dot{x}(t)\,x(t)] = \int_0^\infty \dot{x}\,x e^{-st}\,dt = \dot{x}\int_0^\infty x\,e^{-st}\,dt - \ddot{x}\int\int x\,e^{-st}\,dt\,dt$$

$$+ \dddot{x}\int_0^\infty\int\int x\,e^{-st}\,dt\,dt\,dt - \ldots \tag{3-1}$$

Let: $I_1 = \dot{x}\int_0^\infty x\,e^{-st}\,dt$

Knowing the transform of the term under the integral, this term then yields:

$$I_1 = \dot{x}(0)\;X(s) \tag{3-2}$$

Similiarly allow:

$$I_2 = -\ddot{x}\int_0^\infty\int x\,e^{-st}\,dt\,dt = \ddot{x}(0)\;\frac{X(s)}{s} \tag{3-3}$$

and:

$$I_3 = \dddot{x}\int_0^\infty\int\int x\,e^{-st}\,dt\,dt\,dt = \dddot{x}(0)\,\frac{X(s)}{s^2} \tag{3-4}$$

The summation of these terms gives:

$$\mathcal{L}\,[\dot{x}x] = \sum_{n=1}^{\infty} I_n = [I_1 + I_2 + I_3 + \ldots]$$

$$= X(s)\left[\dot{x}(0) + \frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \ldots\right] \tag{3-5}$$

Simplifying:

$$\mathcal{L}\,[\dot{x}(t)\;x(t)] = s^2\,X(s)\left[-\,\frac{x(0)}{s} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \dots\right]$$

$$= s^2\,X(s)\left[X(s) - \frac{x(0)}{s}\right]$$

$$= s\,X(s)\;[s\,X(s) - x(0)] \tag{3-6}$$

Continuing in this fashion the Laplace Transform for $\dot{x}(t)x^2(t)$ is derived.

$$\mathcal{L}\,[\dot{x}(t)\;x^2(t)] = \int_0^\infty \dot{x}\,x^2 e^{-st}dt = \dot{x}\int_0^\infty x^2 e^{-st}dt - \ddot{x}\int_0^\infty\!\!\int x^2 e^{-st}dt\,dt$$
$$+\,\dddot{x}\int_0^\infty\!\!\int\!\!\int x^2 e^{-st}dt\,dt\,dt - \dots \tag{3-7}$$

Let: $I_1 = \dot{x}\int_0^\infty x^2 e^{-st}dt = \dot{x}\left[x\int_0^\infty x\,e^{-st}\,dt - \dot{x}\int_0^\infty\!\!\int x\,e^{-st}dt\,dt + \dots\right.$

Further let: $J_1 = x\int_0^\infty x\;e^{-st}dt = x(0)\;X(s)$

$$J_2 = -\,\dot{x}\int_0^\infty\!\!\int x\,e^{-st}dt\,dt = \frac{\dot{x}(0)\;X(s)}{s}$$

$$J_3 = \ddot{x}\int_0^\infty\!\!\int\!\!\int x\,e^{-st}dt\,dt\,dt = \frac{\ddot{x}(0)\;X(s)}{s^2}$$

Summing:

$$I_1 = \dot{x}\sum_{n=1}^{\infty} J_n = \dot{x}(0)\;X(s)\left[x(0) + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \dots\right]$$

and after simplifying:

$$I_1 = s\,\dot{x}(0)\,X^2(s) \tag{3-8}$$

Similiarly let:

$$I_2 = -\ddot{x}\iint_0^\infty x^2 e^{-st}\,dt\,dt = -\ddot{x}\left[x\iint_0^\infty x e^{-st}\,dt\,dt - \dot{x}\iiint_0^\infty x e^{-st}\,dt\,dt\,dt + \ldots\right]$$

$$L_1 = x\iint_0^\infty x e^{-st}\,dt\,dt = -\frac{x(0)\,X(s)}{s}$$

$$L_2 = -\dot{x}\iiint_0^\infty x e^{-st}\,dt\,dt\,dt = -\frac{\dot{x}(0)\,X(s)}{s^2}$$

Summing again:

$$I_2 = -\ddot{x}\sum_{n=1}^{\infty} L_n = -\ddot{x}(0)\,X(s)\left[-\frac{x(0)}{s} - \frac{\dot{x}(0)}{s^2} - \frac{\ddot{x}(0)}{s^3} - \ldots\right]$$

$$= \ddot{x}(0)\,X^2(s) \tag{3-9}$$

By similiar arguments obtain:

$$I_3 = \dddot{x}\iiint_0^\infty x^2 e^{-st}\,dt\,dt\,dt = \frac{\dddot{x}(0)\,X^2(s)}{s} \tag{3-10}$$

Combining the above integral terms the transform is then:

$$\mathcal{L}\left[\dot{x}(t)\,x^2(t)\right] = \sum_{n=1}^{\infty} I_n = s^3 X^2(s)\left[-\frac{x(0)}{s} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \ldots\right]$$

$$= s^2 X^2(s)\,\left[s\,X(s) - x(0)\right] \tag{3-11}$$

In similiar fashion the transform for $\dot{x}(t)\,x^3(t)$ may be obtained and expressed as:

$$\mathcal{L}\left[\dot{x}(t)\,x^3(t)\right] = \int_0^\infty \dot{x}\,x^3 e^{-st}\,dt = s^3 X^3(s)\,\left[s\,X(s) - x(0)\right] \tag{3-12}$$

Equations (3-6), (3-11) and (3-12) provide the Laplace Transform for the general form of the function, $\dot{x}(t)\,x^n(t)$.

$$\mathcal{L}\,[\dot{x}(t)\; x^n(t)] = s^n X^n(s) \;\{\mathcal{L}\,[\dot{x}(t)]\} \tag{3-13}$$

B. TRANSFORMS FOR THE PRODUCT OF A FUNCTION AND ITS SECOND DERIVATIVE

Employing again the method of integration by parts to the non-linear function, $\ddot{x}(t)x(t)$, the transform may be expressed as:

$$\mathcal{L}\,[\ddot{x}(t)\,x(t)] = \int_0^\infty \ddot{x}\,x\,e^{-st}\,dt = \ddot{x}\int_0^\infty x\,e^{-st}\,dt - \dddot{x}\int_0^\infty\!\!\int x\,e^{-st}\,dt\,dt + \ldots \tag{3-14}$$

Attacking each of the integrals separately as in Section III.A we let:

$$I_1 = \ddot{x}\int_0^\infty x\,e^{-st}\,dt = \ddot{x}(0)\; X(s) \tag{3-15}$$

$$I_2 = -\dddot{x}\int_0^\infty\!\!\int x\,e^{-st}\,dt\,dt = \frac{\dddot{x}(0)\;X(s)}{s} \tag{3-16}$$

$$I_3 = x^{(4)}\int_0^\infty\!\!\int\!\!\int x\,e^{-st}\,dt\,dt\,dt = \frac{x^{(4)}(0)\;X(s)}{s^2} \tag{3-17}$$

Therefore:

$$\mathcal{L}\,[\ddot{x}(t)\,x(t)] = \sum_{n=1}^{\infty} I_n = \ddot{x}(0)\;X(s) + \frac{\dddot{x}(0)\;X(s)}{s} + \frac{x^{(4)}(0)\;X(s)}{s^2} + \ldots$$

$$= s^3\;X(s)\left[-\frac{x(0)}{s} - \frac{\dot{x}(0)}{s^2} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots\right] \tag{3-18}$$

Collecting terms and simplifying:

$$\mathcal{L}[\ddot{x}(t)\;x(t)] = s\;X(s)\;[s^2\;X(s) - s\,x(0) - \dot{x}(0)] \tag{3-19}$$

The derivation of the Laplace Transform of $\ddot{x}(t)\,x^2(t)$ may be similiarly obtained.

$$\mathcal{L}\,[\ddot{x}(t)\,x^2(t)] = \int_0^\infty \ddot{x}\;x^2\,e^{-st}\,dt = \ddot{x}\int_0^\infty x^2\,e^{-st}\,dt - \dddot{x}\int_0^\infty\!\!\int x^2\,e^{-st}\,dt\,dt + \ldots \tag{3-20}$$

Knowing the transform of $x^2(t)$ again write separately for each of the integrals:

$$I_1 = \ddot{x} \int_0^\infty x^2 e^{-st}\, dt = \ddot{x}(0)\; sX^2(s) \tag{3-21}$$

$$I_2 = -\dddot{x} \iint_0^\infty x^2 e^{-st}\, dt\, dt = \dddot{x}(0)\; X^2(s) \tag{3-22}$$

$$I_3 = x^{(4)} \iiint_0^\infty x^2 e^{-st}\, dt\, dt\, dt = \frac{x^{(4)}(0)\; X^2(s)}{s} \tag{3-23}$$

Summing the above terms the transform becomes:

$$\mathcal{L}[\ddot{x}(t)\, x^2(t)] = \sum_{n=1}^{\infty} I_n = \ddot{x}(0)\, s\, X^2(s) + \dddot{x}(0)\, X^2(s) + \frac{x^{(4)}(0)\; X^2(s)}{s} + \ldots$$

$$= s^4 X^2(s) \left[- \frac{x(0)}{s} - \frac{\dot{x}(0)}{s^2} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots\right] \tag{3-24}$$

$$\mathcal{L}[\ddot{x}(t)\, x^2(t)] = s^2 X^2(s)\; [s^2 X(s) - s\,x(0) - \dot{x}(0)] \tag{3-25}$$

The transform for $\ddot{x}(t)\, x^3(t)$ may be identically expressed as:

$$\mathcal{L}[\ddot{x}(t)\; x^3(t)] = \int_0^\infty \ddot{x}\, x^3 e^{-st}\, dt = \ddot{x} \int_0^\infty x^3 e^{-st}\, dt - \dddot{x} \iint_0^\infty x^3 e^{-st}\, dt\, dt + \ldots \tag{3-26}$$

$$\mathcal{L}[\ddot{x}(t)\; x^3(t)] = s^3 X^3(s)\; [s^2 X(s) - s\,x(0) - \dot{x}(0)] \tag{3-27}$$

Comparison of equations (3-19), (3-25) and (3-27) provides the general form of the transform for these functions.

$$\mathcal{L}[\ddot{x}(t)\; x^n(t)] = s^n X^n(s)\; [s^2 X(s) - s\,x(0) - \dot{x}(0)]$$

$$= s^n X^n(s)\; \{\mathcal{L}\, [\ddot{x}(t)]\} \tag{3-28}$$

C. TRANSFORM OF THE PRODUCT OF POWERS OF A FUNCTION AND ITS DERIVATIVES

Combining the results of the two foregoing sections the general form of the transform of the product of a function raised to the n^{th} power and its various derivatives can be derived. From the derived transforms as expressed in equations (3-13) and (3-28):

$$\mathcal{L}[\dot{x}(t)\, x^n(t)] = s^n X^n(s)\ \{\mathcal{L}[\dot{x}(t)]\}$$

$$\mathcal{L}[\ddot{x}(t)\, x^n(t)] = s^n X^n(s)\ \{\mathcal{L}[\ddot{x}(t)]\}$$

The Laplace Transform of the product of a function raised to the n^{th} power and its m^{th} derivative is therefore:

$$\mathcal{L}[x^{(m)}(t)\, x^n(t)] = s^n X^n(s)\ \{\mathcal{L}[x^{(m)}(t)]\} \tag{3-29}$$

D. THE TRANSFORM OF THE FIRST DERIVATIVE RAISED TO THE n^{th} POWER

From Section II-B, equation (2-5) gives the transform of $\dot{x}(t)$.

$$\mathcal{L}[\dot{x}(t)] = s\,X(s) - x(0)$$

Proceeding with the first derivative squared, $[\dot{x}(t)]^2$, and expressing the transform using integration by parts, the Laplace Transform becomes:

$$\mathcal{L}[(\dot{x})^2] = \int_0^\infty \dot{x}^2\, e^{-st} = \dot{x}\int_0^\infty \dot{x}\, e^{-st}\,dt - \ddot{x}\iint_0^\infty \dot{x}\, e^{-st}\,dt\,dt + \ldots \tag{3-30}$$

Knowing the transform of $\dot{x}(t)$ the terms of the series may be expressed separately.

$$I_1 = \dot{x}\int_0^\infty \dot{x}\, e^{-st}\,dt = \dot{x}(0)[s\,X(s) - x(0)] \tag{3-31}$$

$$I_2 = -\ddot{x}\iint_0^\infty \dot{x}\, e^{-st}\,dt\,dt = \frac{\ddot{x}(0)}{s}\,[s\,X(s) - x(0)] \tag{3-32}$$

$$I_3 = \dddot{x}\int_0^\infty\!\!\iiint \dot{x}\, e^{-st}\, dt\, dt\, dt = \frac{\dddot{x}(0)}{s^2}\,[s\,X(s) - x(0)] \tag{3-33}$$

$$\mathcal{L}\{[\dot{x}(t)]^2\} = \sum_{n=1}^{\infty} I_n = I_1 + I_2 + I_3 + \ldots$$

$$= [s\,X(s) - x(0)]\left[\dot{x}(0) + \frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \ldots\right]$$

$$= [s\,X(s) - x(0)]s^2\left[-\frac{x(0)}{s} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots\right]$$

$$= s^2\,[s\,X(s) - x(0)]\,\left[X(s) - \frac{x(0)}{s}\right]$$

$$= s\,[s\,X(s) - x(0)]^2 \tag{3-34}$$

Using similiar arguments the transform of $[\dot{x}(t)]^3$ may be derived.

$$\mathcal{L}\{\dot{x}(t)]^3\} = \int_0^\infty \dot{x}^3 e^{-st}\, dt = \dot{x}\int_0^\infty \dot{x}^2 e^{-st}\, dt - \ddot{x}\int_0^\infty\!\!\iint \dot{x}^2 e^{-st}\, dt\, dt + \ldots \tag{3-35}$$

Knowing the transform of $[\dot{x}(t)]^2$ from equation (3-34) let:

$$I_1 = \dot{x}\int_0^\infty \dot{x}^2 e^{-st}\, dt = \dot{x}(0)\; s\;[s\,X(s) - x(0)]^2 \tag{3-36}$$

$$I_2 = -\,\ddot{x}\int_0^\infty\!\!\iint \dot{x}^2 e^{-st}\, dt\, dt = \frac{\ddot{x}(0)}{s}\; s\;[s\,X(s) - x(0)]^2 \tag{3-37}$$

$$I_3 = \dddot{x}\int_0^\infty\!\!\iiint \dot{x}^2 e^{-st}\, dt\, dt\, dt = \frac{\dddot{x}(0)}{s^2}\; s\;[s\,X(s) - x(0)]^2 \tag{3-38}$$

Summing the above terms:

$$\mathcal{L}\{[\dot{x}(t)]^3\} = \sum_{n=1}^{\infty} I_1 + I_2 + I_3 + \ldots$$

$$= s\,[s\,X(s) - x(0)]^2\left[\dot{x}(0) + \frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \ldots\right]$$

$$= s^3 [s\,X(s) - x(0)]^2 \left[- \frac{x(0)}{s} + \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots \right]$$

$$= s^2 \; [s\,X(s) - x(0)]^3 \qquad (3\text{-}39)$$

From equations (3-34) and (3-39) the transform of $[\dot{x}(t)]^n$ may be obtained.

$$\mathcal{L}\,\{[\dot{x}(t)]^n\} = s^{n-1}\; [s\,X(s) - x(0)]^n$$

$$= s^{n-1}\; \{\mathcal{L}\,[\dot{x}(t)]\}^n \qquad (3\text{-}40)$$

IV. APPLICATION OF LAPLACE TRANSFORMS TO NON-LINEAR SYSTEMS

A. EXAMPLE OF A FIRST-ORDER NON-LINEAR SYSTEM, EQUATION 1

The first non-linear system to which the derived Laplace Transforms are to be applied is one which has previously been considered by Baycura [Ref. 3] and Brady [Ref. 4] in the initial work that has been accomplished in this area. The differential equation to be considered is of the form:

$$\dot{x} + ax + bx^2 = 0$$

The reasons for using this particular equation, aside from providing some measure of continuity in previous work, are that it represents a stable system and is by far one of the simplest non-linear equations with which to work.

In this initial analysis the author will bring to light several concepts and techniques that utilize Laplace Transforms to derive solutions to various equations and then compare these to solutions obtained from more "conventional" techniques. The general concept will be to express the transform of the unknown variable in a series of terms in s, the solution to which may easily be obtained on the digital computer. The two solutions used for comparison will have been obtained from an analytical method and by computer using Runge-Kutta fourth-order integration methods using Adams-Moulton Predictor Corrector with error check.

Utilizing, therefore, the equation:

$$\dot{x} + ax + bx^2 = 0 \qquad (4-1)$$

and applying the derived Laplace Transforms we obtain:

$$bsX^2(s) + (s + a)X(s) - x(0) = 0 \tag{4-2}$$

Letting $x(0) = c$ then:

$$bsX^2(s) + (s + a)X(s) - c = 0 \tag{4-3}$$

Using the quadratic formula, solve for X(s).

$$X(s) = \frac{s + a}{2bs} \left\{ - 1 \pm \left[1 + \frac{4bcs}{(s+a)^2} \right]^{\frac{1}{2}} \right\} \tag{4-4}$$

The radical may be expanded in a binomial series of the form of:

$$(1+\mu)^n = 1 + n\mu + \frac{n(n-1)\mu^2}{2!} + \frac{n(n-1)(n-2)\mu^3}{3!} + \ldots \tag{4-5}$$

Letting a = 2, b = 1, and x(0) = c = 1 the series expression for X(s) is:

$$X(s) = \frac{1}{s+2} - \frac{s}{(s+2)^3} + \frac{2s^2}{(s+2)^5} - \frac{5s^3}{(s+2)^7} + \frac{14s^4}{(s+2)^9} - \frac{42s^5}{(s+2)^{11}} + \frac{132s^6}{(s+2)^{13}} - \ldots \tag{4-6}$$

To obtain x(t) the inverse transform of the individual terms may be obtained and summed. An alternate and by far a more convenient and simpler method was, however, used in which X(s) was expressed as a ratio of polynomials in s. X(s) then took the form:

$$X(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}$$

A devised computer technique was then used to obtain x(t). This method to find the inverse Laplace Transform by digital computer is thoroughly described in Appendix A.

Using these computer methods a solution to the equation was easily obtained using any desired number of terms of the series. As a result, solutions to the equation were examined terminating with the second to the eighth term in the series.

Several sets of solutions have therefore been derived and are tabulated in Table IV-1. These are an exact solution which is known and obtained through analytical methods [Ref. 5], a computer solution using Runga-Kutta integration methods, and several computer solutions using Laplace Transforms but using varied numbers of terms in the truncated series. The Runga-Kutta solution is presented in order to show its accuracy since this solution will be the basis for comparison in subsequent examples.

TABLE IV-1

SOLUTION TO EQUATION 1

TIME (t)	EXACT SOLUTION	RUNGE-KUTTA SOLUTION	APPROX. 4 TERMS	APPROX. 6 TERMS	APPROX. 8 TERMS
0.0	1.000	1.000	1.000	1.000	1.000
0.2	0.576	0.575	0.579	0.579	0.582
0.4	0.352	0.352	0.367	0.369	0.370
0.6	0.223	0.223	0.247	0.249	0.252
1.0	0.0950	0.0944	0.131	0.101	0.140
1.4	0.0410	0.0414	0.0841	-0.0761	0.0959
1.8	0.0184	0.0183	0.0574	-0.464	0.0791
2.2	0.00823	0.00822	0.0370	-1.231	0.0939
2.6	0.00368	0.00368	0.0216	-2.434	0.189
3.0	0.00165	0.00165	0.0116	-3.969	0.521
3.4	0.000734	0.000735	0.00636	-5.594	1.402
3.8	0.000333	0.000327	0.00416	-7.024	3.246
4.2	0.000153	0.000145	0.00350	-8.028	6.407
4.6	0.0000667	0.0000643	0.00334	-8.482	11.003
5.0	0.0000300	0.0000283	0.00318	-8.387	16.817

In comparing these various solutions, it is noted that the approximate transform solution conforms quite favorably with the exact solution using up through five terms of the series of equation (4-6). Truncating beyond five terms, the series has a tendency to diverge, the direction and magnitude of divergence depending on the number of terms beyond five that are utilized. Since this series should converge for all values of t, it can be concluded that the original derivation of the transform:

$\mathcal{L}[x(t)] = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots$ is invalid for large values of t and is the direct cause of this divergence. For small values of time, t, however, all solutions closely approximate the true solution.

B. EXAMPLE OF A NON-LINEAR EQUATION WITH SQUARE OF DERIVATIVE, EQUATION 2

Considered now is the Froude Equation in its general form:

$$\ddot{x} + 2\zeta\,(1 + \alpha\,\dot{x})\,\dot{x} + x = 0 \tag{4-7}$$

This equation demonstrates non-linear damping, but otherwise behaves in a rather linear fashion. For a stable system select $\zeta = 0.1$, $\alpha = 0.5$. Then equation (4-7) becomes:

$$\ddot{x} + .2\dot{x} + .1(\dot{x})^2 + x = 0 \tag{4-8}$$

Taking transforms we have:

$$s^2X(s) - sx(0) - \dot{x}(0) + .2sX(s) - .2x(0) + .1s^3X^2(s) - .2x(0)s^2X(s)$$
$$+ .1[x(0)]^2 s + X(s) = 0. \tag{4-9}$$

Collecting terms and letting $x(0) = 1$ and $\dot{x}(0) = 0$,

$$.1s^3X^2(s) + [.8s^2 + .2s + 1]X(s) - (.9s + .2) = 0 \tag{4-10}$$

Now solve for X(s) using the quadratic formula and arrange in a suitable form:

$$X(s) = \frac{(.8s^2 + .2s + 1)}{.2s^3} \left\{ -1 \pm \left[1 + \frac{.4s^3 (.9s + .2)}{(.8s^2 + .2s + 1)^2} \right]^{\frac{1}{2}} \right\} \qquad (4\text{-}11)$$

Expanding the term within the radical in a binomial series and after collecting terms and rearranging, X(s) can be expressed in a series.

$$X(s) = \frac{(.9s + .2)}{(.8s^2 + .2s + 1)} - \frac{.1s^3(.9s + .2)^2}{(.8s^2 + .2s + 1)^3} + \frac{.02s^6(.9s + .2)^3}{(.8s^2 + .2s + 1)^5} - \cdots \qquad (4\text{-}12)$$

The inverse Laplace Transform, x(t), is now obtained using the method depicted in Appendix A. The solution, compared with the equation solution using a Runge-Kutta integration method, is displayed in Table IV-2.

The Runge-Kutta (true) solution and the approximate solution with values corresponding to those displayed in Table IV-2 are depicted in graphical form in Figure 1. The true solution is a damped sinusoid. The approximate solution in this case is fairly accurate for small values of t (up to 3 seconds). This conforms with the analysis previously given in Section IV-A.

Introducing appropriate initial conditions at periodic intervals of approximately three seconds, several piecewise but continuing solutions were obtained. This method provided a continuous solution but one that could be obtained over short periods of time. Figure 1 shows that a very accurate solution could be obtained in this manner.

TABLE IV-2

SOLUTION TO EQUATION 2

TIME	RUNGE-KUTTA SOLUTION	TRANS FORM SOLUTION
0.0	1.000	1.000
0.4	0.922	0.899
0.8	0.709	0.735
1.2	0.397	0.466
1.6	0.038	0.121
2.0	-0.311	-0.253
2.4	-0.596	-0.595
2.8	-0.774	-0.769
3.2	-0.823	-0.807
3.6	-0.743	-0.707
4.0	-0.558	-0.493
4.4	-0.308	-0.207
4.8	-0.037	0.054
5.2	0.212	0.205
5.6	0.405	0.400
6.0	0.520	0.520
6.4	0.546	0.557
6.8	0.487	0.515
7.2	0.358	0.407
7.6	0.182	0.249
8.0	-0.012	0.061

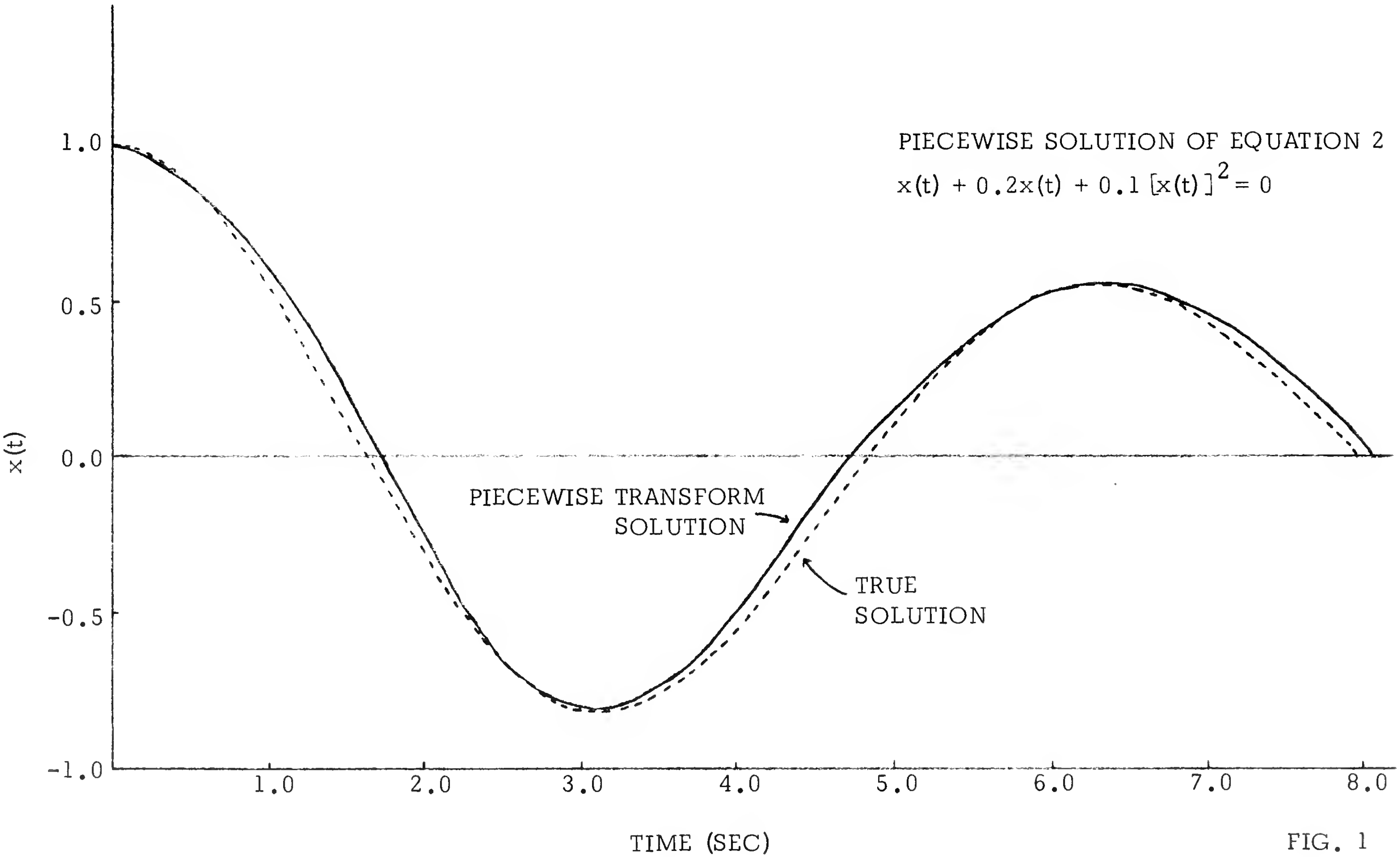
PIECEWISE SOLUTION OF EQUATION 2
x(t) + 0.2x(t) + 0.1 [x(t)]^2 = 0
PIECEWISE TRANSFORM SOLUTION
TRUE SOLUTION
x(t)
1.0
0.5
0.0
-0.5
-1.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
TIME (SEC)

FIG. 1

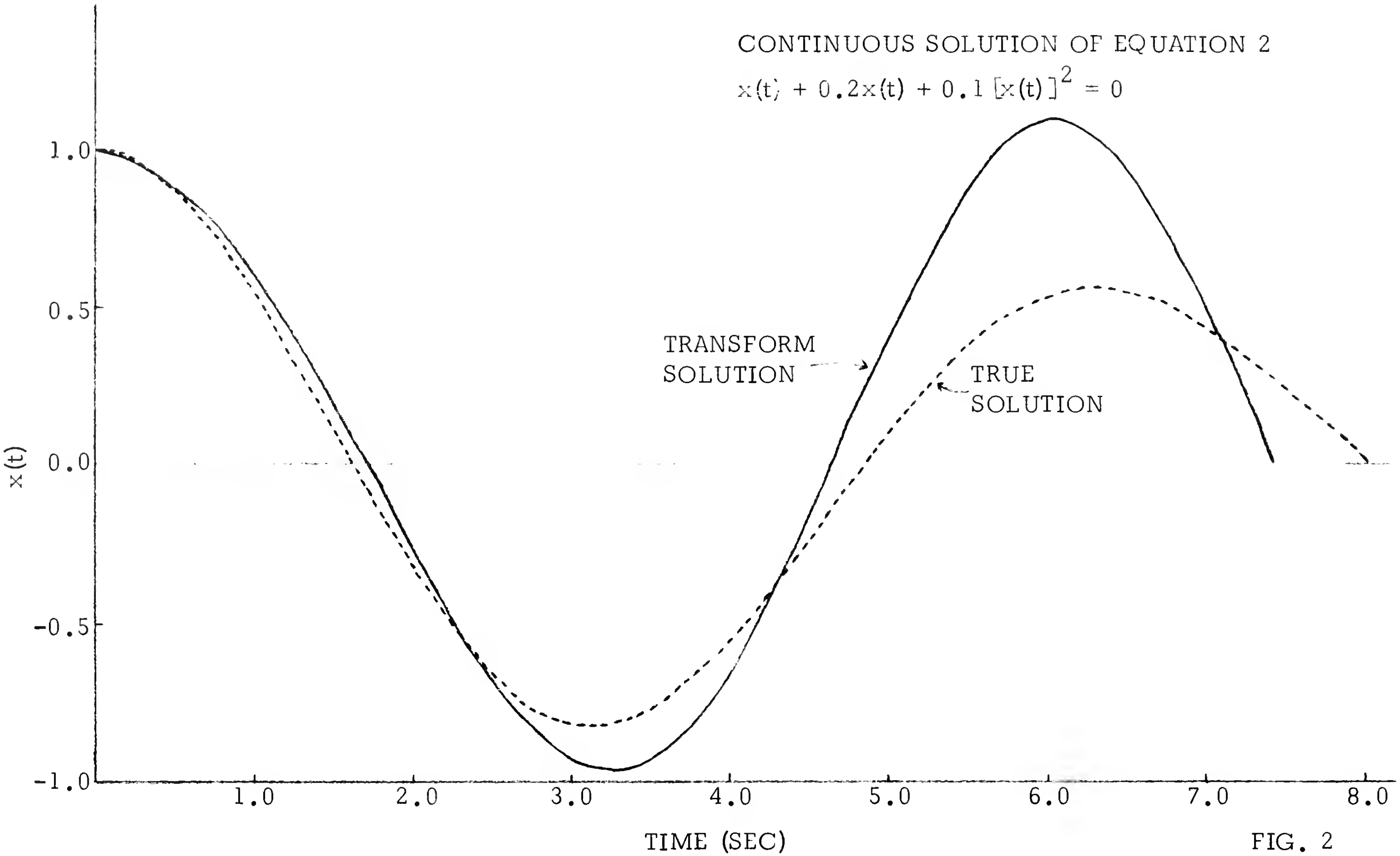
CONTINUOUS SOLUTION OF EQUATION 2
$x(t) + 0.2x(t) + 0.1\,[x(t)]^2 = 0$
TRANSFORM SOLUTION
TRUE SOLUTION
x(t)
1.0
0.5
0.0
-0.5
-1.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
TIME (SEC)

FIG. 2

In contrast to this method Figure 2 depicts a true and approximate solution, the latter being obtained in a continuous fashion from zero to eight seconds. It is noted that prior to three seconds of time the solution is fairly accurate but subsequent to this time begins to diverge and becomes slightly out of phase. The divergence is noted to increase with time.

C. EXAMPLE OF THE PRODUCT OF THE VARIABLE AND ITS DERIVATIVE, EQUATION 3

We next consider the van der Pol Equation which describes various physical situations, the more noted representing an electronic oscillator. The general expression for this equation is:

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + kx = U$$

Where U represents the input and α and k are contants. Consider the system with zero input and with k = 1, then the equation becomes:

$$\ddot{x} - \alpha\dot{x} + \alpha x^2\dot{x} + x = 0 \qquad (4\text{-}12)$$

Apply derived transforms.

$$s^2 X(s) - sx(0) - \dot{x}(0) - \alpha s X(s) + \alpha x(0) + \alpha s^3X^3(s) - \alpha s^2x(0)X^2(s) + X(s) = 0 \qquad (4\text{-}13)$$

After collecting terms the expression becomes:

$$\alpha s^3X^3(s) - \alpha s^2 x(0)X^2 s + [s^2 - \alpha s + 1]X(s) - [sx(0) - \alpha x(0) + \dot{x}(0)] = 0 \qquad (4\text{-}14)$$

Solving this cubic equation for X(s) is no easy task; however, if we assume that the initial conditions for the higher derivative terms are negligible and equally zero then:

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} \tag{4-15}$$

Substituting for X(s) in the cubed term yields:

$$\alpha s\dot{x}(0)X^2(s) + [s^2 - \alpha s + 1]X(s) - [sx(0) - \alpha x(0) + \dot{x}(0)] = 0 \tag{4-16}$$

Let: $x(0) = a$

$\dot{x}(0) = b$

$\dot{x}(0) - \alpha x(0) = c$

Substituting these values in the equation for X(s) a quadratic equation is obtained:

$$\alpha bsX^2(s) + [s^2 - s + 1]X(s) - [as + c] = 0\ . \tag{4-17}$$

Using the quadratic formula now solve for X(s).

$$X(s) = \frac{-(s^2 - \alpha s + 1) \pm [(s^2 - \alpha s + 1)^2 + (4b\alpha s)(as + c)]^{\frac{1}{2}}}{2\alpha bs}$$

$$= \frac{s^2 - \alpha s + 1}{2\alpha bs}\left\{-1 \pm \left[1 + \frac{(4b\alpha s)(as + c)}{(s^2 - \alpha s + 1)^2}\right]\right\} \tag{4-18}$$

Expanding the term under the radical in a binomial series and collecting terms yields:

$$X(s) = \frac{(as + c)}{(s^2 - \alpha s + 1)} - \frac{(b\alpha s)(as + c)^2}{(s^2 - \alpha s + 1)^3} + \frac{2(b\alpha s)^2(as + c)^3}{(s^2 - \alpha s + 1)^5} - \dots \tag{4-19}$$

Now the original equation, (4-11), represents a system having variable damping depending on the displacement x and hence the coefficient of the x term. Allowing:

$\alpha = 1$

$x(0) = a = 1$

$\dot{x}(0) = b = 1$

$\dot{x} - \alpha x(0) = c = 0$

equation (4-19) becomes:

$$X(s) = \frac{s}{(s^2 - s + 1)} - \frac{s^3}{(s^2 - s + 1)^3} + \frac{2s^5}{(s^2 - s + 1)^5} - \ldots . \quad (4\text{-}20)$$

Again solve for x(t) using the devised computer methods. The results are compared with a Runge-Kutta solution in Table IV-3. For α small, as in our example, a number of cycles of oscillation are required before the steady state is achieved. The true solution is shown over a period of one cycle of the sinusoidal solution but during this period the initial growth can be detected commensurate with the slight increase in amplitude with appropriate increase in time.

It should be noted at this time that the derived series representation for X(s) as depicted by equation (4-20) is an unstable system. This fact is borne out by the approximate solution as shown in Table IV-3. It can be generally stated then that the technique adopted here for finding Laplace Transforms is not applicable to systems, such as represented by the van der Pol Equation, which change from a condition of instability to one of stability. The method being investigated will not show the true system but will cause the solution to diverge without bound with increasing time.

TABLE IV-3

SOLUTION TO EQUATION 3

TIME	RUNGE-KUTTA SOLUTION	TRANS FORM SOLUTION
0.0	1.000	1.000
0.2	1.176	1.174
0.4	1.295	1.273
0.6	1.351	1.256
0.8	1.349	1.079
1.2	1.205	0.071
1.6	0.902	-2.049
2.0	0.421	-5.374
2.4	-0.319	-9.583
2.8	-1.261	-13.723
3.2	-1.850	-16.041
3.6	-1.913	-13.938
4.0	-1.744	-4.184
4.4	-1.473	16.526
4.8	-1.109	50.561
5.2	-0.601	97.907
5.6	0.160	154.297
6.0	1.187	209.379
6.4	1.893	245.178
6.8	1.994	235.719
7.2	1.837	148.605

V. CONCLUSION

In the foregoing analysis a logical method was presented by which the Laplace integral was expanded into an infinite series and after integrating by parts the infinite series developed could then be applied to non-linear systems. It was shown that this technique is applicable to most systems or differential equations, the exceptions being systems that are unstable. The limitations restricting the use of this method arise primarily from the complexity of the equation and the inherent difficulty in solving for X(s), the transform of the dependent variable x(t). The procedure used in this paper was to restrict the expression for X(s) to a quadratic equation and to further express the transform in a converging series using a binomial series expansion.

It was observed that in obtaining the inverse of the transform a close approximation to the true equation/system solution could be obtained. This approximation was found to be valid for small values (several seconds) of solution time but for larger values of time a slow digression from the true solution was obtained. This anomaly was attributed to the errors introduced through the use of the transform expressed in a series of terms consisting of the initial conditions of the unknown variable and taking the form of:

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \ldots$$

In the attempt to find a reliable yet uncomplicated method to produce the inverse of the Laplace Transform and hence the solution to the system, a computer program was developed whereby these painstaking computations could then be readily handled on the digital computer. This procedure permitted a more detailed analysis of the problem otherwise unobtainable through ordinary analytical methods.

Finally, since the approximate solution was found to be valid for only a small interval of time a more meaningful comparison with the true solution was necessary. Through the introduction of appropriate initial conditions at periodic intervals, several piecewise yet continuing solutions to an example equation were obtained. This was easily executed with the use of the associated digital program. The solution obtained in this fashion proved very accurate and usable for all time.

APPENDIX A

COMPUTATION OF INVERSE LAPLACE TRANSFORM

The solution of example non-linear equations through application of derived Laplace Transforms is based primarily on computational techniques proposed by Ward and Strum [Ref. 6]. These techniques are based on expressing the system's equations in simultaneous first-order state equations such as:

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\,\underset{\sim}{x}(t) + \underset{\sim}{B}\,\underset{\sim}{u}(t) \tag{A-1}$$

where: $\underset{\sim}{x}(t)$ is the matrix of state variables

and $\underset{\sim}{u}(t)$ is the matrix of system inputs.

Incorporating a time domain solution of these state equations, x(t), computed at any particular time, t, may be then expressed as:

$$\underset{\sim}{x}(t) = \epsilon^{\underset{\sim}{A}t}\,\underset{\sim}{x}(0) + \epsilon^{\underset{\sim}{A}t}\int_0^t \epsilon^{-\underset{\sim}{A}t}\,\underset{\sim}{Bu}\,(\tau)\,d\tau \tag{A-2}$$

A computer program is presented by the authors to solve for x.

With the transform of x given as:

$$X(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n} \tag{A-3}$$

the problem then is to transform equation (A-3) into the form of equation (A-2).

By cross multiplying

$$(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n)\,X(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n \tag{A-4}$$

The right side is then written in matrix form.

$$[s^{n-1} \quad s^{n-2} \quad s^{n-3} \quad \ldots . \; 1] \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \qquad \text{(A-5)}$$

If X(s) resulted from the differential equation:

$x^{(n)} + a_1 x^{(n-1)} + a_2 s^{(n-2)} + \ldots .. + a_n x = 0$ with initial conditions:

$x(0), x^{(1)}(0), \ldots .. x^{(n-1)}(0)$ then the transform of the equation is:

$$s^n X(s) - s^{n-1} x(0) - s^{n-2} x^{(1)}(0) - \ldots ... - s^{\circ} x^{(n-1)}(0)$$

$$+ a_1 [s^{n-1} X(s) - s^{n-2} x(0) - s^{n-3} x^{(1)}(0) - \ldots ... - s^{\circ} x^{(n-2)}(0)]$$

$$+ \ldots ...$$

$$+ a_{n-1} [s X(s) - s^{\circ} x(0)]$$

$$+ a_n X(s) = 0 \qquad \text{(A-6)}$$

Now collecting all I.C. terms and moving them to the right, the right side of the equation then becomes:

$$= [s^{n-1} + a_1 s^{n-2} + a_2 s^{n-3} + \ldots . + a_{n-1}] x(0)$$

$$+ [s^{n-2} + a_1 s^{n-3} + \ldots . a_{n-2}] x^{(1)}(0)$$

$$+ \ldots ..$$

$$+ [1] x^{(n-1)}(0) \qquad \text{(A-7)}$$

Rearranging again into a matrix product:

$$= [s^{n-1} \; s^{n-2} \; \ldots \; s \; 1] \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_1 & 1 & 0 & \ldots & 0 \\ a_2 & a_1 & 1 & \ldots & 0 \\ \cdot & a_2 & \cdot & \ldots & \cdot \\ \cdot & & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & \cdot \quad \cdot & \ldots & \cdot \\ a_{n-1} & a_{n-2} & \cdot \quad \cdot & \ldots & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x^{(1)}(0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^{(n-1)}(0) \end{bmatrix} \qquad (A-8)$$

Now equations (A-5) and (A-8) may be equated so that:

$$\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_1 & 1 & 0 & \ldots & 0 \\ a_2 & a_1 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ a_{n-1} & a_{n-2} & \cdot & \ldots & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x^{(1)}(0) \\ x^{(2)}(0) \\ \cdot \\ \cdot \\ \cdot \\ x^{(n-1)}(0) \end{bmatrix} \qquad (A-9)$$

For easier notation the above matrices may be written as:

$$\underset{\sim}{b} = \underset{\sim}{a}\,\underset{\sim}{x}(0) \qquad (A-10)$$

or: $\underset{\sim}{x}(0) = \underset{\sim}{a}^{-1}\underset{\sim}{b}$ (A-11)

Applying these results to equation (A-2) note that in this case the input, the u matrix, will be zero, leaving but one term for x(t).

$$\underset{\sim}{x}(t) = \epsilon^{\underset{\sim}{A}t}\,\underset{\sim}{x}(0) \qquad (A-12)$$

where A is a matrix of constants as given in equation (A-1) and is of the form:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_n & -a_{n-1} & \cdot & \cdots & -a_1 \end{bmatrix}$$

Knowing $\underset{\sim}{a}$ in equation (A-11) $\underset{\sim}{x}(0)$ may be easily obtained with a computer utilizing a library program to obtain the inverse of a matrix and then multiplying the two resulting matrices.

APPENDIX B

LIST OF LAPLACE TRANSFORMS

1. $\mathcal{L}[x(t)] = X(s)$

2. $\mathcal{L}[x^2(t)] = s\,X^2(s)$

3. $\mathcal{L}[x^3(t)] = s^2\,X^3(s)$

4. $\mathcal{L}[x^n(t)] = s^{n-1}\,X^n(s)$

5. $\mathcal{L}[\dot{x}(t)] = s\,X(s) - x(0)$

6. $\mathcal{L}[\ddot{x}(t)] = s^2\,X(s) - s\,x(0) - \dot{x}(0)$

7. $\mathcal{L}[x^{(m)}(t)] = s^m X(s) - s^{m-1}\,x(0) - s^{m-2}\,\dot{x}(0) - \ldots - s\,x^{(m-2)}(0)$
   $-x^{m-1}(0)$

8. $\mathcal{L}[\dot{x}(t)] = s\,X(s) - x(0)$

9. $\mathcal{L}[\{\dot{x}(t)\}^2] = s[s\,X(s) - x(0)]^2$

10. $\mathcal{L}[\{\dot{x}(t)\}^n] = s^{n-1}[s\,X(s) - x(0)]^n = s^{n-1}\,\{\mathcal{L}[\dot{x}(t)\}^n$

11. $\mathcal{L}[\{x^{(m)}(t)\}^n] = s^{n-1}\,\{\mathcal{L}[x^{(m)}(t)]\}^n$

12. $\mathcal{L}[\dot{x}(t)\,x(t)] = s\,X(s)\,[s\,X(s) - x(0)]$

13. $\mathcal{L}[\dot{x}(t)\,x^2(t)] = s^2X^2(s)\,[s\,X(s) - x(0)]$

14. $\mathcal{L}[\dot{x}(t)\,x^n(t)] = s^nX^n(s)\,\{\mathcal{L}[\dot{x}(t)]\}$

15. $\mathcal{L}[\ddot{x}(t)\,x(t)] = s\,X(s)\,[s^2\,X(s) - s\,x(0) - \dot{x}(0)]$

16. $\mathcal{L}[\ddot{x}(t)\,x^2(t)] = s^2X^2\,s[s^2X(s) - s\,x(0) - \dot{x}(0)]$

17. $\mathcal{L}[\ddot{x}(t)\,x^n(t)] = s^n\,X^n(s)\,\{\mathcal{L}[\ddot{x}(t)]\}$

18. $\mathcal{L}[x^{(m)}(t)\,x^n(t)] = s^n\,X^n(s)\,\{\mathcal{L}[x^{(m)}(t)]\}$

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13. ABSTRACT

The use of Laplace Transforms has long been one of the primary methods utilized in solving linear differential equations but recently the extension and application of this method to the solution of basic non-linear systems has been proposed. This latter technique is presented and expanded to include various non-linear functions. The applicability of this method to systems of varied form and complexity is then explored and the validity of the derived solution investigated.

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